Multicritical points and crossover mediating the strong violation of universality: Wang-Landau determinations in the random-bond $d=2$ Blume-Capel model

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The effects of bond randomness on the phase diagram and critical behavior of the square lattice ferromagnetic Blume-Capel model are discussed. The system is studied in both the pure and disordered versions by the same efficient two-stage Wang-Landau method for many values of the crystal field, restricted here in the second-order phase-transition regime of the pure model. For the random-bond version several disorder strengths are considered. We present phase diagram points of both pure and random versions and for a particular disorder strength we locate the emergence of the enhancement of ferromagnetic order observed in an earlier study in the ex-first-order regime. The critical properties of the pure model are contrasted and compared to those of the random model. Accepting, for the weak random version, the assumption of the double-logarithmic scenario for the specific heat we attempt to estimate the range of universality between the pure and random-bond models. The behavior of the strong disorder regime is also discussed and a rather complex and yet not fully understood behavior is observed. It is pointed out that this complexity is related to the ground-state structure of the random-bond version.

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I. INTRODUCTION

The effect of quenched randomness on the equilibrium and dynamic properties of macroscopic systems is a subject of great theoretical and practical interest. It is well known that quenched bond randomness may produce drastic changes on phase transitions depending on the type of the transition [1–6]. Thus, symmetry-breaking first-order transitions are converted to second-order phase transitions by infinitesimal bond randomness for spatial dimensionality $d=2$ [3,4] and by bond randomness beyond a threshold strength in $d>2$ [4], as indicated by general arguments [5] and in some cases by rigorous mathematical work [3]. In particular, this rounding effect of first-order transitions has now been rigorously established in a unified way in low dimensions ($d\leq 2$) including a large variety of types of randomness in classical and quantum spin systems [7].

Historically, the effects of disorder on phase transitions have been studied in two extreme cases, i.e., in the limits of weak and strong (near the percolation point) disorder. The first important conjecture, known today as the Harris criterion [1], relates the value of the specific-heat exponent α in a continuous transition with the expected effects of uncorrelated weak disorder in ferromagnets. According to the Harris criterion, for continuous phase transitions with a negative exponent α, the introduction of weak randomness is expected to be an irrelevant field and the disordered system to remain in the same universality class. On the other hand, the weakly disordered system is expected to be in a different universality class in the case of a pure system having a positive exponent α. Pure systems with a zero specific-heat exponent ($\alpha=0$) are marginal cases of Harris criterion (since the criterion does not give any information) and their study, upon the introduction of disorder, has been of particular interest. The paradigmatic model of the marginal case is, of course, the general random two-dimensional (2D) Ising model (random site, random bond, and bond diluted) and this model has been extensively investigated and debated [8–24]. Several recent studies, both analytical (renormalization group and conformal field theories) and numerical [mainly Monte Carlo (MC) simulations] devoted to this model, have provided very strong evidence in favor of the so-called logarithmic corrections' scenario. According to this, the effect of infinitesimal disorder gives rise to a marginal irrelevance of randomness and besides logarithmic corrections, the critical exponents maintain their 2D Ising values. In particular, the specific heat is expected to slowly diverge with a double-logarithmic dependence [11–14]. Here, we should mention that there is not full agreement in the literature and a different scenario predicts a negative specific heat exponent α leading to a saturating behavior [16], with a corresponding correlation length exponent $\nu\geq 2/d$ [25].

In general, a unitary and rigorous physical description of critical phenomena in disordered systems still lacks and certainly, lacking such a description, the study of further models for which there is a general agreement in the behavior of the corresponding pure cases is very important. Historically, such a suitable candidate for testing the above predictions, that has been also quite extensively studied, is the general 2D q state Potts model [17,26–30]. This model includes the Ising model ($q=2$), cases of a pure system having continuous transitions with a positive exponent α ($q=3,4$) and also the large q cases ($q>4$) for which one could observe and try to classify the above-mentioned softening of first-order transitions in 2D models. Another similarly interesting candidate, not yet as much studied in the random-bond version, is the 2D Blume-Capel (BC) model [31,32]. We may note here that most of the existing literature on the BC model with randomness concerns randomness applied to the crystal field and/or spin glass exchange interactions [33–36]. As it is well

known, the pure version of the BC model undergoes an Ising-like continuous phase transition to an ordered ferromagnetic phase as the temperature is lowered for crystal-field couplings less than a tricritical value and a first-order transition for larger values of the crystal-field coupling. Therefore, this model provides also the opportunity to study two different and very interesting topics of the above described effects of disorder in critical phenomena, namely, the double-logarithmic scenario for the specific heat in the regime where the 2D BC model is in the same universality class with the 2D random-bond Ising model and also the softening of the transition in the first-order regime. Recently the present authors [37] have considered this model and provided strong numerical evidence clarifying two of the above mentioned effects induced in 2D systems by bond randomness. By implementing a two-stage Wang-Landau (WL) approach [37–43], we presented essentially exact information on the 2D BC model under quenched bond randomness. In this investigation, we found dramatically different critical behaviors of the second-order phase transitions emerging from the first- and second-order regimes of the pure BC model and since, these second-order transitions were found to have different critical exponents, our study indicated an interesting strong violation of universality [37]. Namely, different sets of critical exponents on two segments of the same critical line appeared to describe the two regimes: still-second-order and ex-first-order.

In this paper, we extend our earlier work [37], by implementing essentially the same two-stage WL approach (Sec. II) and try to give a more complete picture by concentrating in the weak (still-second-order) regime and simulate the model for several disorder strengths and many values of the crystal-field coupling. The above statement means that, effectively we will restrict our study to moderate values of the crystal field and moderate values of the disorder, intending to observe the frontier between the weak- and the strong-disorder universality classes from the disappearance of the expected 2D random Ising universality class behavior. Thus, in Sec. III we will produce phase diagram points for the random-bond model but also for the pure model, reporting for the pure case a comparison with existing estimates in the literature. More generally, in carrying out this project we have also considered the pure 2D BC model for several values of the crystal field, in the second-order regime, observing its finite-size scaling (FSS) behavior. Sec. IV presents such a comparative study between random and pure models concerning the behavior of all thermodynamic parameters used in the traditional FSS analysis of MC data. This study enables us to observe some peculiarities of the pure model, due to the onset of tricriticality, and compare them with the corresponding behavior of the random model. Furthermore, we try to focus, understand and shed light to the extent of universality of the random-bond 2D BC model with the corresponding random-bond Ising model, for which the scenario of logarithmic corrections seems to be the strongest option in the current literature [24]. The prediction of the range of such universality is far from trivial and the two regimes (weak and strong) have much dissimilarities which are also reflected in the ground-state structure, as further discussed below. The attempt to estimate the range of the above-mentioned universality is accomplished by the idea which assumes the truth of the double-logarithmic scenario for the specific heat in a suitable restricted range. This is presented in Sec. V together with some further crucial observations concerning the behavior of the strong disorder regime, i.e., the regime where the Ising class universality does not apply and the system has a rather complex and yet not understood behavior. We have also indicated in this section, the increased difficulty of analysis, innate to the physical problem, under strong randomness. Our conclusions are summarized in Sec. VI.

II. DEFINITION OF THE MODELS AND THE TWO-STAGE WANG-LANDAU APPROACH

A. Pure and random-bond Blume-Capel models

The (pure) BC model [31,32] is defined by the Hamiltonian

$$H_p = -J\sum_{\langle ij\rangle} s_i s_j + \Delta\sum_i s_i^2, \tag{1}$$

where the spin variables s_i take on the values $-1, 0$, or $+1$, $\langle ij\rangle$ indicates summation over all nearest-neighbor pairs of sites, and $J>0$ is the ferromagnetic exchange interaction. The parameter Δ is known as the crystal-field coupling and to fix the temperature scale we set $J=1$ and $k_B=1$. As it is well known, this model has been analyzed, besides the original mean-field theory [31,32], by a variety of approximations and numerical approaches. These include the real space renormalization group, MC simulations, and MC renormalization-group calculations [44], ϵ-expansion renormalization groups [45], high- and low-temperature series calculations [46], a phenomenological FSS analysis using a strip geometry [47,48], and, finally, a recent two-parameter WL sampling in rather small lattices of linear sizes $L\leq 16$ [49]. As mentioned already in the introduction the phase diagram of the model consists of a segment of continuous Ising-like transitions at high temperatures and low values of the crystal field which ends at a tricritical point, where it is joined with a second segment of first-order transitions between (Δ_t, T_t) and $(\Delta=2,\ T=0)$.

The model given by Eq. (1) is studied here on the square lattice and will be referred to as the pure BC model. However, our main focus, on the other hand, is the case with bond disorder given by the bimodal distribution

$$P(J_{ij}) = \frac{1}{2}[\delta(J_{ij}-J_1) + \delta(J_{ij}-J_2)];$$

$$\frac{J_1+J_2}{2} = 1; \quad J_1 > J_2 > 0; \quad r = \frac{J_2}{J_1}, \tag{2}$$

so that r reflects the strength of the bond randomness. The present study concerns only the case of 50%/50% weak/strong bonds, as indicated above by the bimodal distribution in Eq. (2), this practice follows the traditional bond disorder implementation commonly used in the case of the square lattice Ising model [8]. Other choices of bond randomness have been commented in Sec. V. The resulting quenched

disordered (random-bond) version of the Hamiltonian defined in Eq. (1) reads now as

$$H = -\sum_{\langle ij \rangle} J_{ij} s_i s_j + \Delta \sum_i s_i^2. \quad (3)$$

A first comparative study between the two versions (pure and random) of the 2D BC model has been already presented in our earlier paper [37] for two values of the crystal-field coupling corresponding to the second-order ($\Delta=1$) and first-order ($\Delta=1.975$) regimes of the pure model where for the random version the disorder strength $r=0.75/1.25=0.6$ was chosen in both cases. In the next Sections our study on the 2d BC model is extended to several values of the crystal field and disorder (listed in the table of Sec. III). Details of our simulations are summarized in the next Section together with an up to date brief sketch of our entropic scheme.

B. Outline of our implementation of the Wang-Landau approach

In the last few years we have used an entropic sampling implementation of the WL algorithm [38,39] to study some simple [40,41], but also some more complex systems [8,42,43]. One basic ingredient of this implementation is a suitable restriction of the energy subspace for the implementation of the WL algorithm. This was originally termed as the critical minimum energy subspace (CrMES) restriction [40,41] and it can be carried out in many alternative ways, the simplest being that of observing the finite-size behavior of the tails of the energy probability density function (e-pdf) of the system [41]. Complications that may arise in random systems can be easily accounted for by various simple modifications that take into account possible oscillations in the e-pdf and expected sample-to-sample fluctuations of individual disorder realizations. In our recent papers [8,37,43], we have presented details of various sophisticated routes for the identification of the appropriate energy subspace (E_1, E_2) for the entropic sampling of each random realization. In estimating the appropriate subspace from a chosen pseudocritical temperature one should be careful to account for the shift behavior of other important pseudocritical temperatures and extend the subspace appropriately from both low- and high-energy sides in order to achieve an accurate estimation of all finite-size anomalies. Of course, taking the union of the corresponding subspaces insures accuracy for the temperature region of all studied pseudocritical temperatures.

The up to date version of our implementation uses a combination of several stages of the WL process. First, we carry out a starting (or preliminary) multirange (multi-R) stage, in a very wide energy subspace. This preliminary stage is performed up to a certain level of the WL random walk. The WL refinement is $G(E) \rightarrow f * G(E)$, where $G(E)$ is the density of states (DOS) and we follow the usual modification factor adjustment $f_{j+1}=\sqrt{f_j}$ and $f_1=e$. The preliminary stage may consist of the levels: $j=1,\ldots,j=18$ and to improve accuracy the process may be repeated several times. However, in repeating the preliminary process and in order to be efficient, we use only the levels $j=13,\ldots,18$ after the first attempt, using as starting DOS the one obtained in the first random walk at the level $j=12$. From our experience, this practice is almost equivalent of simulating the same number of independent WL random walks. Also in our recent studies, we have found out that is much more efficient and accurate to loosen up the originally applied very strict flatness criteria [40,41]. Thus, a variable flatness process starting at the first levels with a very loose flatness criteria and assuming at the level $j=18$ the original strict flatness criteria is now days used. After the above described preliminary multi-R stage, in the wide energy subspace, one can proceed in a safe identification of the appropriate energy subspace using one or more alternatives outlined in Refs. [40,41]. In random systems, where one needs to simulate many disorder realizations, it is also possible and advisable to avoid the identification of the appropriate energy subspace separately for each disorder realization by extrapolating from smaller lattices and/or by prediction from preliminary runs on small numbers of disorder realizations. In any case, the appropriate subspaces should be defined with sufficient tolerances. In our implementation we use such advance information to proceed in the next stages of the entropic sampling.

The process continues in two further stages (two-stage process), using now mainly high iteration levels, where the modification factor is very close to unity and there is not any significant violation of the detailed balance condition during the WL process. These two stages are suitable for the accumulation of histogram data (for instance energy-magnetization histograms), which can be used for an accurate entropic calculation of nonthermal thermodynamic parameters, such us the order parameter and its susceptibility [41]. In the first (high-level) stage, we follow again a repeated several times (typically $\sim 5-10$) multi-R WL approach, carried out now only in the restricted energy subspace. The WL levels may be now chosen as $j=18,19,20$ and as an appropriate starting DOS for the corresponding starting level the average DOS of the preliminary stage at the starting level may be used. Finally, the second (high-level) stage is applied in the refinement WL levels $j=j_i,\ldots,j_i+3$ (typically $j_i=21$), where we usually test both an one-range (one-R) or a multi-R approach with large energy intervals. In the case of the one-R approach we have found very convenient and in most cases more accurate to follow the Belardinelli and Pereyra [50] adjustment of the WL modification factor according to the rule $\ln f \sim t^{-1}$. Finally, it should be also noted that by applying in our scheme a separate accumulation of histogram data in the starting multi-R stage (in the wide energy subspace) offers the opportunity to inspect the behavior of all basic thermodynamic functions in an also wide temperature range and not only in the neighborhood of the finite-size anomalies. The approximation outside the dominant energy subspace is not of the same accuracy with that of the restricted dominant energy subspace but is good enough for the observation of the general behavior and provides also a route of inspecting the degree of approximation.

The above described numerical approach was used to estimate the properties of a large number of 100 bond disorder realizations, for lattice sizes $L=20-100$ for all crystal fields and disorder strengths used in this paper, with the exception of the case $\Delta=1.5$ and $r=0.5/1.5$, where 500 disorder realizations were simulated for lattice sizes $L=20-140$. For ref-

erence and contrast, the pure system's properties were also obtained by the same implementation, simulating in each case at least 30 independent runs. We close this outline of our numerical scheme with some comments concerning statistical errors and disorder averaging. Even for the larger lattice size studied here ($L=100$) and depending on the thermodynamic parameter, the statistical errors of the WL method were found to be of reasonable magnitude and in some cases to be of the order of the symbol sizes or even smaller. This was true for both the pure version and the individual random-bond realizations. However, since the nature of the present study is qualitative, not aiming to an accurate exponent estimation, the WL errors will not be presented in our figure illustrations. We also note that, for the random-bond version mainly the averages over the disorder realizations, denoted as $[\ldots]_{av}$, will be considered in the text and their finite-size anomalies, denoted as $[\ldots]^{*}_{av}$, will be used in our FSS attempts. Due to very large sample-to-sample fluctuations, mean values of individual maxima ($[\ldots^{*}]_{av}$) have not been used in this study except for illustrative purposes, as in the case $\Delta=1.5$, $r=0.5/1.5$ presented in our last Section.

III. PHASE DIAGRAMS: PURE AND RANDOM-BOND 2D BLUME-CAPEL MODELS

This Section presents, as mentioned in the introduction, phase diagram points for the pure and random-bond models and in the case of the pure model compare the corresponding phase diagram points with the existing estimates in the literature. This gives also the opportunity to observe the reliability of our numerical approach. Following the practice of our earlier paper [37], we estimate phase diagram points by fitting our data to the expected power-law shift behavior $T=T_c+bL^{-1/\nu}$ of several pseudocritical temperatures.

The traditionally used specific heat and magnetic susceptibility peaks, as well as, the peaks corresponding to the following logarithmic derivatives of the powers $n=1,2,4$ of the order parameter with respect to the inverse temperature $K=1/T$ [51],

$$\frac{\partial \ln\langle M^n\rangle}{\partial K}=\frac{\langle M^n H\rangle}{\langle M^n\rangle}-\langle H\rangle, \tag{4}$$

and the peak corresponding to the absolute order-parameter derivative

$$\frac{\partial\langle|M|\rangle}{\partial K}=\langle|M|H\rangle-\langle|M|\rangle\langle H\rangle, \tag{5}$$

will be implemented for a simultaneous fitting attempt.

Such simultaneous fitting attempts are presented in Fig. 1. In particular Fig. 1(a) presents the shift behavior for the random-bond 2D BC model in the weak disorder case $r=0.9/1.1$ at the value $\Delta=1.5$ and Fig. 1(b) illustrates the simultaneous fitting attempted for the pure model at $\Delta=1.95$, further discussed at the end of Sec. IV B. As noted already, the data fitted for the random version of the model are only those of the pseudocritical temperatures of the peaks of the averaged over disorder thermodynamic parameters, as

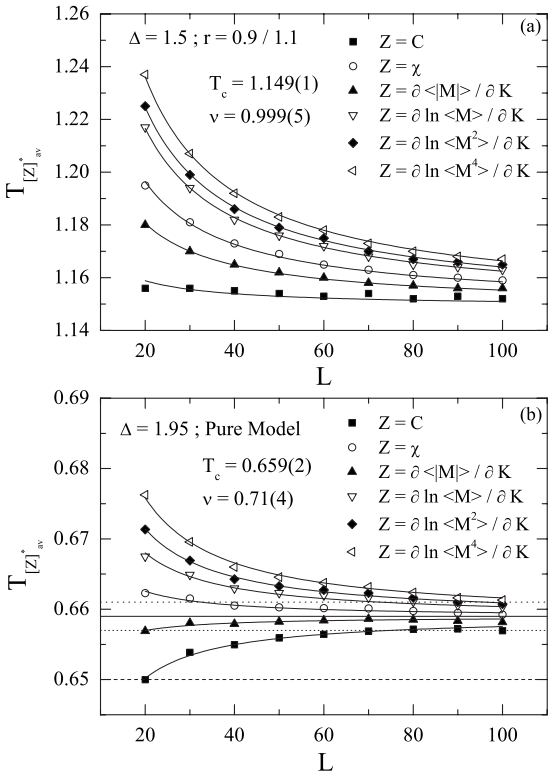

FIG. 1. (a) Simultaneous fitting of the six pseudocritical temperature defined in the text for the random-bond version for $\Delta=1.5$ and $r=0.9/1.1$. The fitting range is $L=20-100$. (b) The same as above simultaneous fitting of the six pseudocritical temperatures now for the pure model at $\Delta=1.95$. The solid line shows our estimate of the critical temperature $T_c=0.659$ (with the dotted lines indicating its errors barriers) and the dashed line the estimate $T_c=0.650$ given in Ref. [48].

indicated in the figure, where the asterisk denotes the peak of the averaged over disorder parameter $[\ldots]^{*}_{av}$. The alternative route of using averages of individual sample parameters gives almost identical estimates. However, in cases of strong lack of self-averaging, very large sample-to-sample fluctuations may be present. For simplicity, in all fitting attempts, the whole range $L=20-100$ has been used. Following this practice for the pure and random version of the 2D BC several phase diagram points were produced. Figure 2 shows the resulting phase diagrams using our phase diagram points for the pure 2D BC model and its random-bond version for the disorder strength $r=0.75/1.25=0.6$. The tricritical point shown is taken from the estimate given by Beale $(\Delta_t,T_t)=[1.9655(10),0.610(5)]$ [48]. For the disorder strength $r=0.75/1.25$ the points of the phase diagram were chosen with the intension to be able to approximately locate the emergence of the enhancement of ferromagnetic order observed in our earlier study [37] in the ex-first-order regime at $\Delta=1.975$, where we found a considerable increase in the critical temperature by $\sim 9\%$. A microscopic explanation of this phenomenon, based on the preference $s_i=\pm 1$ states to the strong-coupling connectivity sites has been given in Ref. [37] (see also Ref. [52]), as a microsegregation process, due to quenched bond randomness, that evolves continuously within the ferromagnetic and paramagnetic phases. The inset

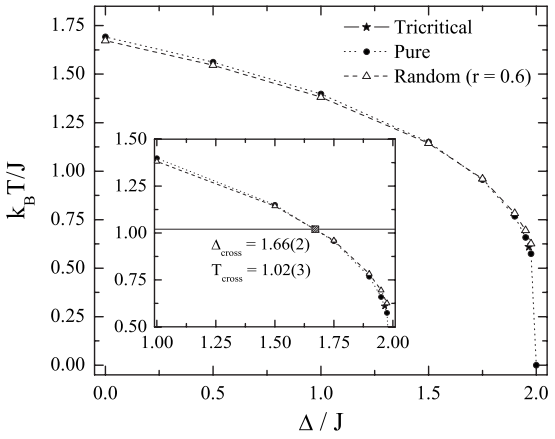

FIG. 2. Phase diagrams of the pure 2D BC model and its random version at the disorder strength $r=0.75/1.25$. The inset illustrates the crossing of the phase boundaries including an approximate estimate of the crossing point.

of Fig. 2 shows that phase diagram of the random version crosses the diagram of the pure model at approximately the crossing point $(\Delta_{cross}, T_{cross})=[1.66(2), 1.02(3)]$, well before the tricritical point. The approximation of this point was obtained by interpolation using the four points of the phase diagram in the range $\Delta=1.0-1.9$.

Table I summarizes the phase diagram points obtained in this paper, by the above described traditional FFS method for both the pure and random version of the 2D BC model for disorder strengths $r=0.9/1.1$, $r=0.75/1.25$, and $r=0.6/1.4$, together with corresponding phase diagram points given by Beale [48] for the pure model. Note that, in Silva *et al.* [49] one can find an analogous table for the pure 2D BC model including the phase diagram points given by Beale [48] and the points produced in their two-parametric WL sampling

TABLE I. Transition temperatures of the pure and random-bond 2D BC model obtained in this paper. Second column from Ref. [48], third and last entries of third and fifth columns from Ref. [37].

Δ/J	k_BT/J				
	Pure		Random		
	Ref. [48]		$r=0.9/1.1$	$r=0.75/1.25$	$r=0.6/1.4$
0	1.695	1.693(3)		1.674(2)	
0.5	1.567	1.564(3)		1.547(2)	
1	1.398	1.398(2)		1.381(1)	
1.2					1.277(3)
1.4					1.184(3)
1.5	1.150	1.151(1)	1.149(1)	1.144(2)	1.131(2)
1.6			1.084(1)		1.071(3)
1.7			1.005(1)		
1.75		0.958(1)		0.960(2)	
1.8			0.908(1)		0.917(3)
1.9		0.769(1)	0.774(2)	0.786(4)	
1.95	0.650	0.659(2)		0.702(3)	
1.975		0.574(2)		0.626(2)	

[49]. It can be seen that there is an excellent coincidence of our points with those of Beale [48]. The points of Beale [48] are based on the very accurate phenomenological FSS scheme using a strip geometry [47], whereas our points are obtained via the present simultaneous FSS analysis, based on lattices with linear sizes $L=20-100$. The points of Silva *et al.* [49] are based in much smaller lattices of the two-parameter WL sampling (linear sizes $L\leq 16$). However, in the case $\Delta=1.95$ our phase diagram point does not agree, within errors, with that of Beale and for this reason our estimation with a rather generous error bars (shown also on the panel) has been illustrated in Fig. 1(b). The rest of Table I contains our estimates for the random-bond versions. One may note from this table that for small values of Δ for instance $\Delta=1.5$ the corresponding critical temperatures decrease as the disorder becomes stronger (compare the three cases of disorder: $r=0.9/1.1$, $r=0.75/1.25$, and $r=0.6/1.4$ at this value of the crystal field). On the other hand, for $\Delta=1.9$ the trend is reversed as can be seen by comparing the corresponding three cases of disorder. Apparently this is a kind of reflection of the phenomenon of the enhancement of ferromagnetic order which appears to influence the geometry of the critical surface for the 2D random-bond BC model.

IV. PHASE TRANSITIONS OF THE PURE AND RANDOM-BOND 2D BLUME-CAPEL MODELS

A. Strong violation of universality: The ex-first-order regime

As pointed out in the introduction, in our recent investigation of the 2d random-bond BC model [37] we found dramatically different critical behaviors of the second-order phase transitions emerging from the first- and second-order regimes of the pure model. Namely, different sets of critical exponents on two segments of the same critical line appeared to describe the two regimes: the still-second-order and ex-first-order regimes. The study in Ref. [37] was carried out for two values of the crystal-field coupling corresponding to the second-order ($\Delta=1$) and first-order ($\Delta=1.975$) regimes of the pure model and for the random version the disorder strength $r=0.75/1.25$ was chosen in both cases. The strong violation of universality observed appeared to be the result of the softening of the first-order transition due to bond randomness. Specifically, it was concluded that the new strong-disorder universality class is well described by a correlation length exponent in the range $\nu=1.30(6)-1.35(5)$, and exponent ratios γ/ν and β/ν very close to the Ising values 1.75 and 0.125, respectively [37]. The above weak universality [16,53,54] seems to be valid between the Ising-like continuous transitions of the random-bond 2D BC model for small values of Δ ($\Delta=1.0$ in Ref. [37]) and continuous transitions belonging to the strong-disorder universality class.

Therefore, the strong-disorder universality class may be characterized by the above distinct value of the correlation length exponent and a strong saturation of the specific heat. Qualitatively this saturating behavior is quite instructive and for illustrative reasons is reproduced here in Fig. 3. This figure contrasts, at $\Delta=1.975$, the specific heat's finite-size behavior of the pure 2D BC model (first-order regime) and two disordered cases corresponding to disorder strengths r

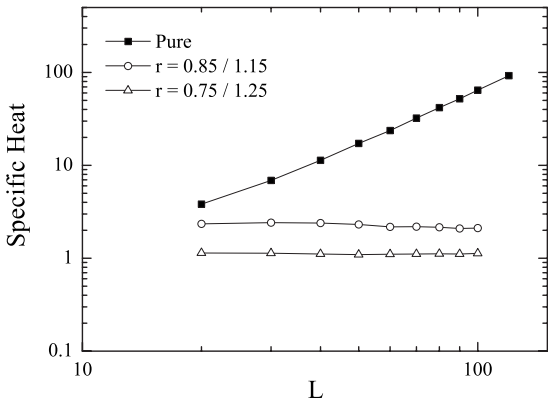

FIG. 3. Behavior of the random-bond 2D BC model at $\Delta=1.975$ from Ref. [37]. Illustration of the divergence of the specific heat of the pure model (first-order regime) and the clear saturation of the specific heat for the random-bond (open symbols) 2D BC model for two disorder strengths in a log-log scale.

$=0.85/1.15$ and $r=0.75/1.25$. The saturation of the specific heat is very clear in both cases of the disorder strength.

It is of interest to point out here that these findings for the strong-disorder universality class appear to be fully compatible with the classification of phase transitions in disordered systems proposed recently by Wu [55]. According to this classification the strong-disorder transition is expected to be inhomogeneous and percolative with an expected exponent of the order $\nu=1.34$ [56]. Furthermore, it has been suggested to us by Wu [57], that the strong lack of self-averaging of this transition stems from the above properties. This violation of self-averaging, together with the strong finite-size effects, makes the systematic MC approach of the strong-disorder regime very demanding, if not impractical. On the other hand, the weak regime (or Ising universality regime) suffers a much weaker lack of self-averaging, by at least a factor of ~ 12 [37], and a smooth behavior is observed at moderate lattice sizes. Thus, aiming here to observe, even approximately, the extent of the involved universality classes we carried out our study at moderate values of the crystal field and disorder, and found a behavior quite convincing from which the frontier of the strong universality class can be estimated by observing the disappearance of the expected 2D random Ising universality class.

B. Pure and random-bond 2D BC model: Range of universality with the 2D Ising model

Let us now proceed with the analysis of our numerical data for the disorder strengths and crystal fields given in Table I and observe and contrast their FSS behavior with that of the pure model.

Starting this comparative study with the FSS of the specific heat maxima (using for the random-bond version at the strength $r=0.75/1.25$ the corresponding quantity averaged over disorder, i.e., $[C]^*_{av}$), we present in Fig. 4 fitting attempts for the same range of Δ for the pure model [Fig. 4(a)] and the random-bond version [Fig. 4(b)]. As indicated by the scales in the x axis and the functions in the corresponding panels, the expected Ising logarithmic divergence has been

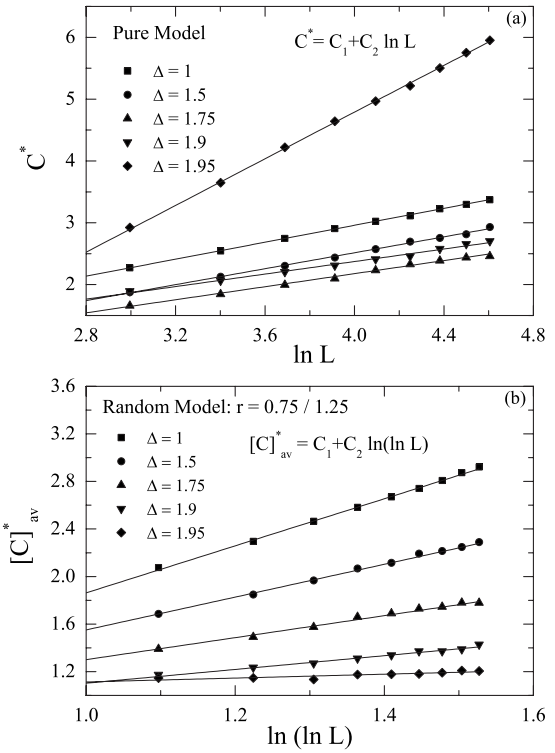

FIG. 4. FSS of the specific heat maxima for the pure and random-bond 2D BC model at the same values of Δ. (a) Pure model: illustration of linear fittings assuming the expected Ising logarithmic divergence. (b) Random model: illustration of linear fittings assuming a double-logarithmic divergence. Note the steady fall of the double-logarithmic amplitude C_2.

assumed for the pure model $C^*=C_1+C_2\ln L$, whereas the double-logarithmic divergence $[C]^*_{av}=C_1+C_2\ln(\ln L)$ has been assumed for the random version. Although, it is very difficult to irrefutably distinguish between a double-logarithmic divergence and a very weak power-law divergence, the theoretically well-grounded double-logarithmic scenario applies very well [37] and this fact can be observed also now for more values of Δ in Fig. 4(b). There are some further features one can observe from Fig. 4. First, for the pure model, and in the range $\Delta>1.9$, we observe a sudden change in the behavior of the specific heat peaks as we approach the tricritical point which is apparently a strong crossover effect. For the random version and the same values of Δ no such strong effects are noticeable and most probably the general softening effects of bond randomness extends also to the expected crossover phenomena between the two different universality classes of the random 2d BC model. From Fig. 4(b) the slopes of the double-logarithmic fittings appear to obey a rather sensible decreasing tendency from which we try in the next Section to locate the frontier of strong-disorder universality class.

A second interesting comparison follows now in Fig. 5, where again Fig. 5(a) presents the FSS of the susceptibility maxima for the pure model, whereas Fig. 5(b) corresponds to the random-bond version at $r=0.75/1.25$. The influence of the exceptionally large fluctuations in the order parameter of the pure model, as we approach the tricritical point, is now

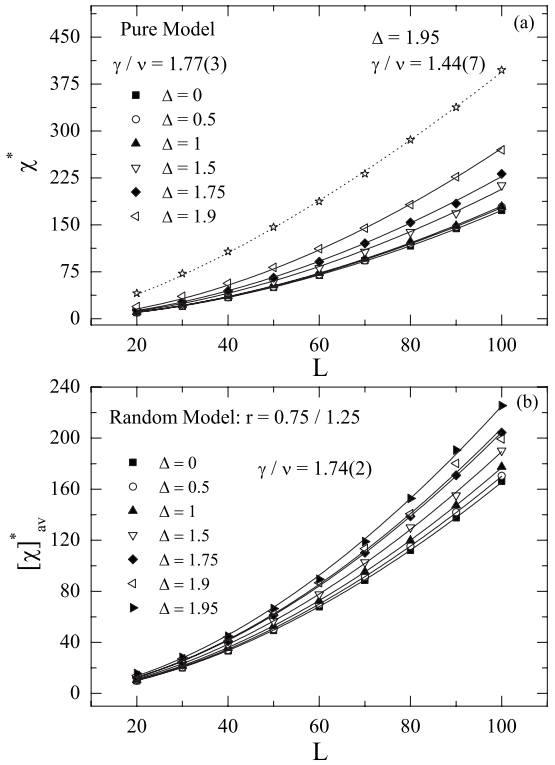

FIG. 5. FSS behavior of the susceptibility maxima for the pure and random-bond 2D BC model at the same values of Δ. (a) Pure model: simultaneous fitting to a simple power law for the first six values of Δ and a separate fitting close to the tricritical point for Δ=1.95. (b) Random model: simultaneous fitting of the averaged susceptibility peaks for all values of Δ. Note the better behavior for the random model and the improved estimation of the exponent ratio γ/ν.

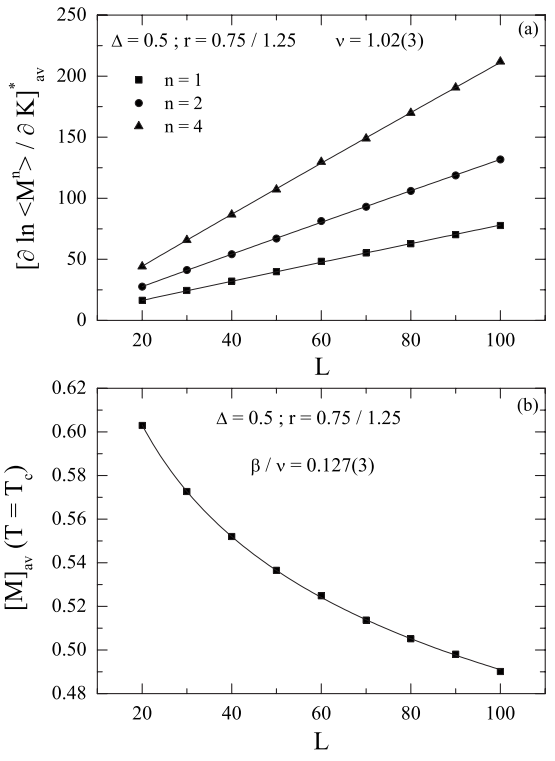

FIG. 6. (a) FSS analysis $\sim L^{1/\nu}$ of the three logarithmic derivatives ($n=1,2,4$) of the order parameter with respect to temperature for a particular crystal field $\Delta=0.5$ and disorder strength $r=0.75/1.25$. (b) Traditional FSS analysis $\sim L^{-\beta/\nu}$ of the order parameter at the estimated critical temperature for the same value of crystal field and disorder strength, as in panel (a).

reflected in the effective exponent ratio γ/ν, estimated by the simple power law $\chi^* \sim L^{\gamma/\nu}$. These large fluctuations are a known peculiarity of the pure model near tricriticality [48]. The effective value of the exponent is now closer to the expected value at the tricritical point $\gamma/\nu=1.5$ [48] than to the value of the Ising universality class $\gamma/\nu=1.75$. For this reason a separate power-law fitting has been applied for the value $\Delta=1.95$ in Fig. 5(a). Note here that in Fig. 5(a) the simultaneous fitting is applied to the first six values of $\Delta=0-1.9$, whereas in Fig. 5(b) all values of $\Delta=0-1.95$ are used. However, even so, the comparison of the two panels of Fig. 5, points out the much better limiting behavior of the random version toward the expected Ising value $\gamma/\nu=1.75$. Therefore, we may convincingly suggest that, for moderate values of crystal field and disorder, the Ising universality scenario (with possible logarithmic corrections) is well obtained in the disordered case.

Figures 6 and 7 illustrate further alternative routes, used commonly in traditional FSS analysis, that provide clear evidence to the above suggestion namely that the weak disorder version belong to the 2D Ising class for suitable moderate values of disorder and crystal-field coupling.

Noteworthy is the fact that the estimation of the critical exponents via the traditional FSS, such as that shown in Figs. 5–7 for the random version, yields estimates very close to the expected values, i.e., the exponents of the 2D Ising model. On the other hand, as pointed also earlier, for the pure model, as we approach the tricritical point ($\Delta=1.95$) the effective exponents remind more those expected at the tricritical point than those of the 2D Ising model. This is particularly true for the correlation length's exponent estimated in Fig. 1(b) by

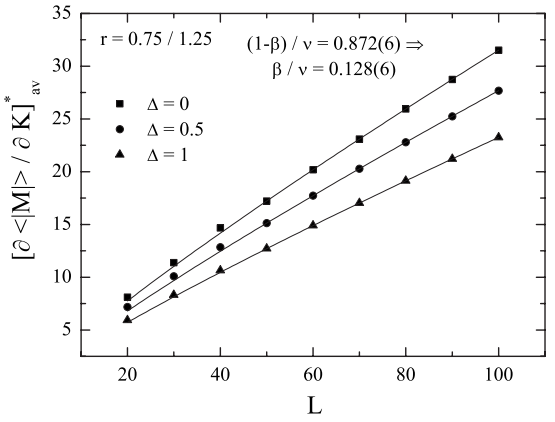

FIG. 7. Simultaneous fitting to a simple power law $\sim L^{(1-\beta)/\nu}$ of the averaged peaks corresponding to the absolute order-parameter derivative for three values of and disorder strength $r=0.6$. The fitting provides an estimate of the exponent $(1-\beta)/\nu$ and as indicated in the panel an alternative estimate for the exponent ratio β/ν, by assuming $\nu=1$.

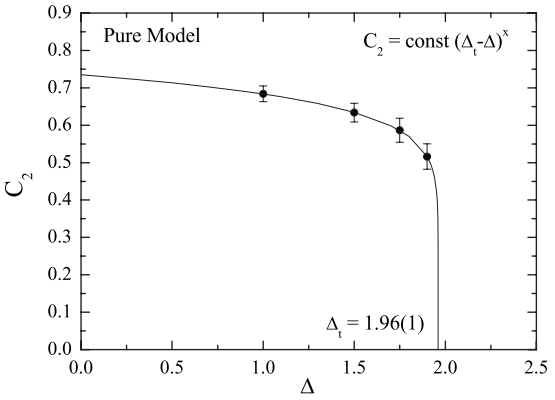

FIG. 8. An approximate estimation of the tricritical value of the crystal field by fitting the decreasing logarithmic amplitudes C_2 of the pure model at suitable values of the crystal-field coupling to a suitable power law shown in the figure.

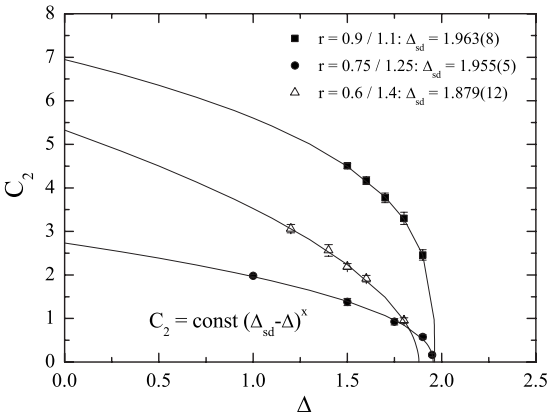

FIG. 9. Estimation of multicritical values of the crystal field, where the 2D random-bond (BC) Ising universality class meets the strong-disorder universality class. The decreasing double-logarithmic amplitudes C_2 of the random version have been fitted to a power law shown and the estimates for the three disorder strengths are shown in the panel.

the simultaneous fitting of the six pseudocritical temperatures. As can be seem from this figure the estimate for the exponent ν, in the case $\Delta=1.95$, is closer to the expected tricritical value $40/77=0.519\cdots$ [48], than to the 2D Ising value $\nu=1$.

V. MULTICRITICAL POINTS AND THE STRONG DISORDER REGIME

A. Estimation of multicritical points

From Fig. 4(a) one can observe the expected Ising logarithmic divergence of the specific-heat maxima. Avoiding the value $\Delta=1.95$, which suffers from strong crossover effects, we attempted to estimate the tricritical value of the crystal field by fitting the decreasing logarithmic amplitudes C_2 to a suitable power law, as shown in Fig. 8. This may appear a naive or questionable idea, since the behavior of specific heat data is the Achilles' heel of FSS analysis. Yet, Fig. 8 shows that besides the large fluctuations (errors) in logarithmic amplitudes C_2, one could approximately estimate the tricritical crystal field $\Delta_t \approx 1.96(1)$, as shown in the panel of Fig. 8.

From Fig. 4(b) we observe again that, the specific heat maxima, corresponding now to the random-bond model at the disorder strength $r=0.75/1.25$, are better-matched to the double-logarithmic divergence than the corresponding specific-heat maxima of the pure model to a simple logarithmic divergence. Therefore, it appears realistic to try to obtain the multicritical point (more precisely the value of the crystal field Δ_{sd} where we expect the emergence of strong-disorder regime at a given disorder strength), where the 2D random-bond (BC) Ising universality class meets the strong-disorder universality class, by fitting the decreasing double-logarithmic amplitudes to a similar power law.

Figure 9 presents now these fittings and also the estimated multicritical values of the crystal field Δ_{sd} for the three disorder strengths considered in this paper. The statistical errors of the corresponding double-logarithmic amplitudes C_2 are seen to be quite smaller, compared to the corresponding amplitudes of the pure model, and the estimated values of the values of the multicritical field, shown in the panel of Fig. 9, appear more convincing. From their general trend we observe that as we increase the disorder strength the frontier of the strong-disorder universality class moves to lower values of the crystal field, namely: $[r=0.9/1.1,\ \Delta_{sd}=1.963(8)]$, $[r=0.75/1.25,\ \Delta_{sd}=1.955(5)]$, and $[r=0.6/1.4,\ \Delta_{sd}=1.879(12)]$. This behavior seems sensible and in our opinion reflects the competition between the ferromagnetic random interactions with the crystal field, giving a kind of destabilization of the usual Ising-like ferromagnetic order. A similar ground-state reflection of this competition is a ground-state structure of unsaturated ferromagnetic ground states discussed in our recent paper [58]. Conversely, as disorder strength is decreased, Δ_{sd} approaches Δ_t, as expected.

B. Strong disorder regime: The case $\Delta=1.5$, $r=0.5/1.5$ and general observations

As pointed out earlier, the strong lack of self-averaging, together with possible strong finite-size effects, make the MC approach to the strong-disorder regime a very difficult task. The self-averaging properties along the two segments (ex-first-order and still-second-order) of the critical line were observed and discussed in Ref. [37].

It was shown in this paper that the usual finite-size measure [59] of relative variance $R_X=V_X/[X]^2_{av}$, where $V_X=[X^2]_{av}-[X]^2_{av}$ (and $X=\chi^*$ is the susceptibility maxima), exhibits lack of self-averaging in both cases of marginal second-order transition and the transition in the strong-disorder or ex-first-order regime. In particular, it was shown [37] that the case studied ($\Delta=1.975$) of the ex-first-order segment gives a much larger effect when compared with the still-second-order regime at $\Delta=1$ and the same disorder strength $r=0.75/1.25$ by a factor of ~ 12. Therefore, phase diagram points very close to the frontier of the strong-disorder regime may be the worst cases to study, because, besides the very strong lack of self-averaging one may also expect large finite-size and crossover effects. However, such cases are elucidatory not only for observing the intrinsic dif-

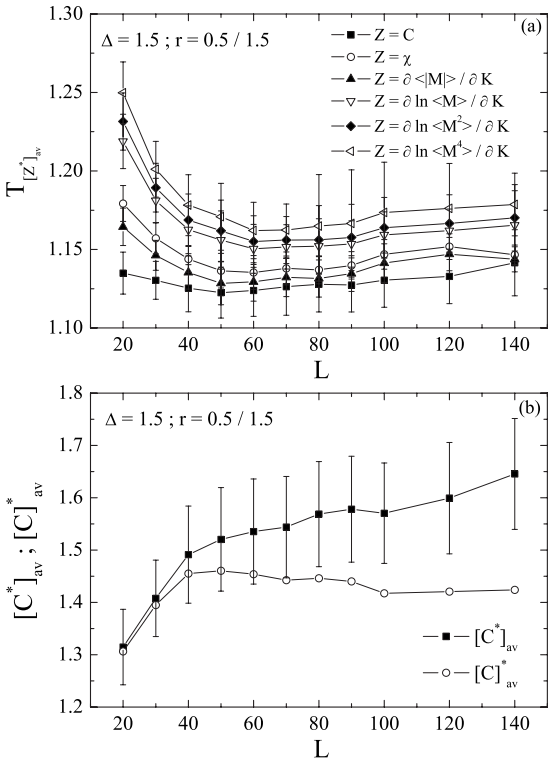

FIG. 10. Finite-size behavior for the case $\Delta=1.5$ and $r=0.5/1.5$. (a) Behavior of the averaged pseudocritical temperatures, corresponding to individual maxima, with an illustration of the huge sample-to-sample fluctuations. (b) Behavior of the maxima of the averaged specific-heat curves ($[C]^*_{av}$) and the average of individual maxima ($[C^*]_{av}$) with their large sample-to-sample fluctuations.

ficulties but also for giving us the opportunity to reflect on possible links with basic properties of the system as for instance the ground-state structure.

Thus, Fig. 10 illustrates two important characteristics of the case $\Delta=1.5$ and $r=0.5/1.5$. In particular, Fig. 10(a) shows the huge sample-to-sample fluctuations in all the pseudocritical temperatures used in this paper. The simulation here was extended to larger lattices ($L=20-140$) and the number of realizations studied was 500, to be compared with 100 realizations studied in previous Sections (weak regime). Besides the enormous fluctuations, the number of 500 realizations looks quite insufficient and strong finite-size effects, reflected as expected in the shift behavior of the system, produce a completely unsettled behavior. Figure 10(b) illustrates an unusual deviating behavior between the finite-size behavior of the maxima of the averaged specific-heat curves ($[C]^*_{av}$) and the finite-size behavior of the average of individual maxima ($[C^*]_{av}$), which as shown suffer large sample-to-sample fluctuations. The behavior here resembles in many aspects the well known and still challenging specific heat behavior of the three-dimensional random-field Ising model [42].

However, a very clear tendency of $[C]^*_{av}$ for a saturating behavior is observed in Fig. 10(b) and we may speculate that this saturating behavior is the correct asymptotic behavior for both maxima shown in Fig. 10(b), although the behavior of $[C^*]_{av}$ will settle down only in very large lattice sizes,

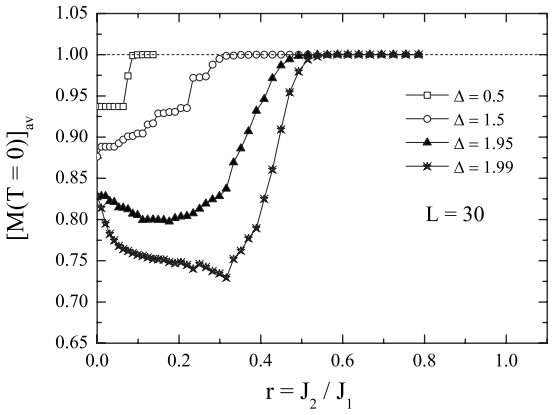

FIG. 11. Ground-state behavior of the order parameter of the 2D random-bond BC model versus r for various values of Δ averaged over 250 disorder realizations.

when the influence of the strong finite-size effects on the individual maxima will diminish.

The above illustrations open the possibility that the case $\Delta=1.5$ and $r=0.5/1.5$ is very close to the frontier of the strong-universality class. This is in accordance with the trend observed in the previous Section and can be further supported by reproducing here some aspects of the ground-state structure of the 2D random-bond BC model. From Fig. 11, reproduced here from Ref. [58], one can see that approximately at this point ($\Delta=1.5$, $r=0.5/1.5$), the system departs from the ferromagnetic ground state and an unsaturated ground state is produced, which is further enhanced with vacant sites ($s_i=0$) as we increase the disorder strength. In the presence of bond randomness the competition between the ferromagnetic interactions with the crystal-field results in a destabilization of the ferromagnetic ground state. Depending on the realization, weak clusters exist in $T=0$ and their points are frozen in the $s_i=0$ state. This is an interesting subject, which is presently under further consideration in both 2D and three dimension (3D) by the present authors. The behavior illustrated in Fig. 10 appears now as a consequence of the onset of the unsaturated ground state at ($\Delta=1.5$, $r=0.5/1.5$) which is thus related with the critical behavior of the 2D random-bond BC model. The presented calculation of the ground states has be carried out in polynomially bounded computing time by mapping the system into a network and searching for a minimum cut by using a maximum flow algorithm (see for instance Ref. [60]) and can be easily extended to large lattices and also to the 3D BC random-bond model.

The ground-state structure as observed from Fig. 11, reflects some aspects of the complexity of the strong disorder regime. For moderate disorder strengths and values of the crystal field, we have large plateaus describing the saturated ferromagnetic ground state, starting from the limit of the pure model and extending to the strong disorder regime. The illustrated characteristic departure from the ferromagnetic ground state is solely due to the competition between the random ferromagnetic interactions with the crystal field. This behavior has been also verified in other random-bond implementations. In the case of a 25%/75% weak/strong bonds

implementation much larger plateaus appear, whereas in the case of a 75%/25% weak/strong bonds implementation much smaller plateaus appear, as should be expected. The problems discussed earlier in this section, and in particular the illustrated behavior for $(\Delta=1.5,\ r=0.5/1.5)$, may be expected in other cases of randomness implementations at the points where the corresponding systems departs from the saturated ferromagnetic ground state. The increased difficulty of analysis is intrinsic to the physical problem under strong randomness and the strong finite-size effects are due to the presence of the created vacancies. However, percolation effects do not enter the problem, since both strong and weak bonds are nonzero ferromagnetic and do not short the lattice (except in the extreme strong disorder limit $r=0$). All bonds on the lattice are present and reinforce the connectivity. Thus, the same quantitative results are obtained with other ratios of strong and weak bonds.

VI. CONCLUSIONS

By carrying out an extensive two-stage Wang-Landau entropic sampling of both the pure and the random-bond 2D Blume-Capel model we have produced phase diagram points for several disorder strengths. Also for the pure model we found an excellent coincidence of our points with those of Beale [48]. For a particular disorder strength $(r=0.75/1.25)$ we found that, as a result of the enhancement of ferromagnetic order, the phase diagram of the random version crosses that of the pure model at approximately the point $(\Delta_{cross},T_{cross})=[1.66(2),1.02(3)]$, well before the tricritical point.

The critical properties of the pure model were compared and contrasted to those of the random model and for moderate values of the crystal field and disorder we found that the Ising universality scenario (with possible logarithmic corrections) is well obtained in the case of the random version. Furthermore, accepting in this range of couplings the assumption of the double-logarithmic scenario for the specific heat, we estimated multicritical points, where the 2D random-bond (BC) Ising universality class meets the strong-disorder universality class.

The behavior of the strong disorder regime was also critically discussed. The case $\Delta=1.5$ and $r=0.5/1.5$ was extensively studied and based on the observed behavior and on the ground-state observations we suggested that this case, most likely, lies on the frontier between the weak- and strong-disorder universality classes, suffering exceptionally strong finite-size effects and a strong lack of self-averaging.

In conclusion, the present paper investigated some difficult and important aspects of the random-bond 2D Blume-Capel model. It has been pointed out that the behavior of this system is very interesting and in particular the strong-disorder regime may include many further challenges and open problems that, at the moment, are not fully understood. In our opinion, this is a rather complex subject deserving further research.

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